EN3: Introduction to Engineering and Statics

Division of Engineering
Brown University

7. Equilibrium

Engineers need to manage forces in structures and machines. There most common problems that engineers face are:

1. To design a system that will apply a set of forces to an object. For example, many manufacturing processes (milling, cutting, injection molding, etc) need to apply large forces to the material being processed; and engineers have to design machines capable of applying these forces. Vehicle design is a second example. An aircraft in flight must be designed carefully to balance gravitational and aerodynamic forces, to ensure that the aircraft remains controllable. Similarly, a car's engine and transmission must be designed to exert whatever forces are necessary to overcome air resistance, rolling resistance, etc while the car travels.
2. To design a structure or machine that can support a given set of external forces for an extended period of time without failure. Examples range from structural problems, to the design of artificial joints to design of micromachines;

To manage forces, engineers need to be able to answer two questions: (1) What forces are required to make a structure move or deform in some prescribed way?; and (2) What *internal forces* are induced in a component by external loading? A structure or machine fails when these internal forces become larger than the material's strength.

Surprisingly, forces in most engineering systems can be understood using the principle of *static equilibrium*. There are really only three main situations where statics doesn't work, (because large accelerations occur)

1. Anything involving impact (crashing vehicles, exploding bombs, etc)
2. Anything that travels along a curved path (an airplane in a turn, cornering vehicles)
3. Systems that contain rapidly rotating parts;
4. Forces induced by *vibrations*.

A static analysis always solves the same problem: given a set of known forces acting on a structure or machine, calculate a set of unknown forces. The unknown forces could be internal forces, or some subset of unknown external forces.

The unknown forces are always calculated by solving *equilibrium equations* for the system.

7.1 Definition

Four and a half simple concepts are required to understand the concept of static equilibrium

1. If a structure is stationary it is in static equilibrium

2. If a machine moves at constant speed along a straight line without rotation it is in static equilibrium

3. The resultant force acting on a structure or machine that is in static equilibrium is zero

4. The resultant moment acting on a structure or machine that is in static equilibrium is zero.

4.5 It doesn't matter what point you choose to take moments about –in a statics problem it's zero about any point. But beware! When you start solving dynamics problems, you will need to take moments about special points to apply equations of motion (usually the center of mass).

That's it! There's nothing more to statics. We can all go home now…

But wait! How about a few examples and applications before you go? We have to do *something* to earn your tuition dollars…

7.2 Examples of the use of static equilibrium to calculate unknown forces

When solving statics problems, we'll always follow the steps below:

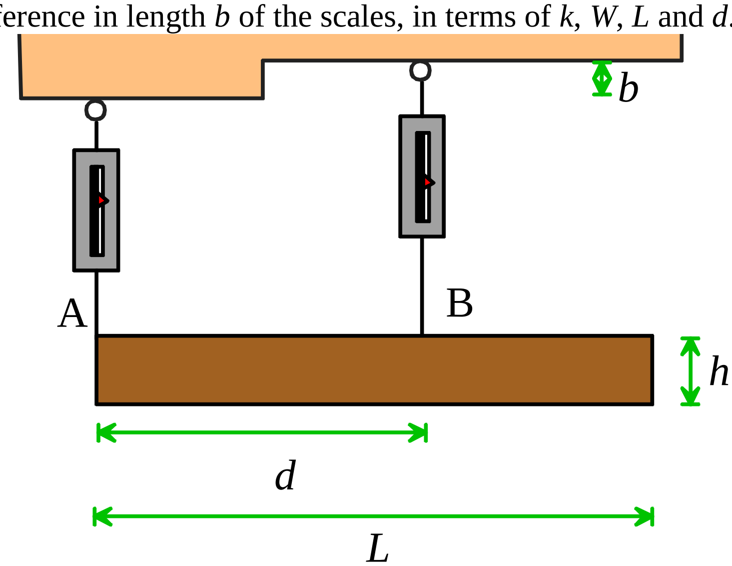
1. Draw a clear picture showing the forces and moments acting on the object(s) of interest. It's important to show the *positions* of the forces correctly;
2. Introduce an appropriate basis to be used for all vector calculations
3. Write down the forces acting on the system (introduce variables to describe unknown forces)
4. Write down the position of the forces
5. Find the moment of the all forces about any convenient point (you must use the same point for each force).
6. Write down any pure moments or torques acting on the system
7. Find the resultant force **F**
8. Find the resultant moment **M**
9. Set **F=0** and **M=0**.
10. If you can, then solve the equations for unknown quantities of interest (often forces, but the unknowns could be other things too, as we shall see).
11. If you have too many unknown forces and not enough equations at this point, you need to look for more equations. These may come from (a) Force balance for other components; (b) Force laws (eg spring law, buoyancy force law, gravity law, etc); (c) geometry.
12. When you have enough equations, solve them.

So let's try some problems and see how this works.

Example 1: A beam with weight W is suspended by two spring-scales, as shown in the picture below.

(i) Find an expression for the force reading on each spring scale in terms of W, L and d.

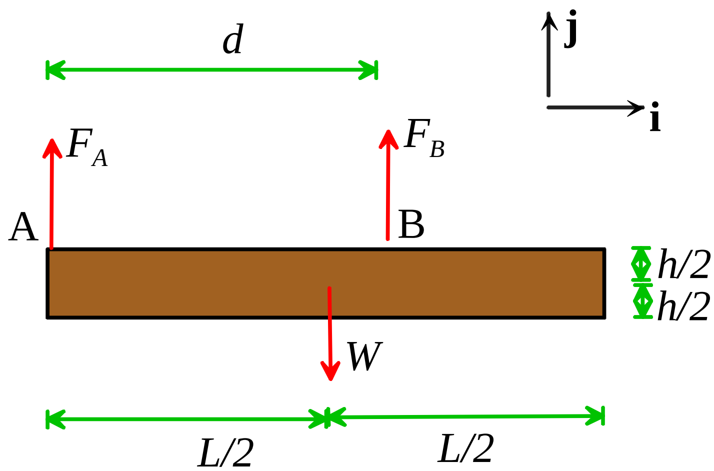
(ii) If the spring scales both have stiffness k and un-stretched length a, find an expression for the difference in length b of the scales, in terms of k, W, L and d.



This is a standard equilibrium problem. We know the weight force acting on the beam. We don't know the forces exerted on the beam by the springs. However, we *do* know that the beam isn't moving and therefore must be in static equilibrium. We can try to calculate the unknown spring forces by writing down equilibrium equations for the beam.

To solve the problem, we follow the steps in our recipe.

(1, 2) Here's the picture showing the forces; as well as a basis for our vectors. We know the weight acts at the center of mass, and can look up the position of the center of mass in the table provided in section 2.1. We also know the spring scales will pull on the beam along the line of the scale, and show the forces accordingly.



Now we proceed to steps 3-9. It's convenient to assemble this information in a *force and moment balance table* as shown below.

Force and moment balance table for beam ABC. Origin at A.			
Force/Moment	Position	Force	Moment about origin
Weight	$(L/2)\mathbf{i}-(h/2)\mathbf{j}$ $(L/2)\mathbf{i}-(h/2)\mathbf{j}$	$-W\mathbf{j}$ $-W\mathbf{j}$	$-(L/2)W\ \mathbf{k}$ $-(L/2)W\ \mathbf{k}$
Force at A	$\mathbf{0}$	$F_A\mathbf{j}$ $F_A\mathbf{j}$	$\mathbf{0}$
Force at B	$d\mathbf{i}$ $d\mathbf{i}$	$F_B\mathbf{j}$ $F_B\mathbf{j}$	$dF_B\mathbf{k}$ $dF_B\mathbf{k}$
Sum (=**0**)		$(F_A+F_B-W)\mathbf{j}$ $(F_A+F_B-W)\mathbf{j}$	$(dF_B-LW/2)\mathbf{k}$ $(dF_B-LW/2)\mathbf{k}$

Note that you can use any of our short-cuts to compute the moments, if you wish. For a statics problem, it doesn't matter what point you choose for the origin. But beware – when you move on to dynamics, you will need to write down dynamical equations of motion by taking moments about special points…

Finally, step 10 - we collect and (try to) solve the governing equations. We get the equations by setting the **i**, **j** and **k** components of resultant force and moment to zero individually. For the problem at hand, we have only two equations

$$F_A+F_B-W=0 \quad F_A+F_B-W=0$$

$$dF_B-LW/2=0 \quad dF_B-LW/2=0$$

We have two equations and two unknown forces, so we are ready to solve them. If, at this point, we had more unknowns than equations, we'd have to look for other principles to apply in order to provide additional equations for the unknowns.

The equations are easily solved for the unknown forces F_A F_A and F_B F_B

$$F_A = W\left(1-\frac{L}{2d}\right) F_A = W\left(1-\frac{L}{2d}\right)$$

$$F_B = W\frac{L}{2d} \qquad F_B = W\frac{L}{2d}$$

This answers part (i) of our problem.

To answer part (ii), we have to look for additional principles to apply – we can't get any more information from statics. Elementary geometry tells us that *b* is the difference in length of the two spring scales. Next, we recall that the length of each spring scale can be computed from the force using the force-extension relation for a spring as

$$k(l_A - a) = F_A k(l_A - a) = F_A$$

$$k(l_B - a) = F_B k(l_B - a) = F_B$$

Therefore, the difference in length is given by

$$b = l_B - l_A = (F_B - F_A)/k b = l_B - l_A = (F_B - F_A)/k$$

$$= \frac{W}{k}\left(\frac{L}{d}-1\right) \qquad = \frac{W}{k}\left(\frac{L}{d}-1\right)$$

giving the solution to part (ii).

If you've taken courses in physics in high school, this procedure may look cumbersome and unnecessary to you. Bear with me. It's true that simple problems like this one can be solved in your head. But the objective of this discussion is to develop a systematic approach that can be used to solve 3D problems involving complex engineering systems, containing tens or even hundreds of unknown forces. You can get completely lost if you try to solve these using an *ad-hoc* procedure.

Example 2

The figure shows a force transducer mounted to the wheel of a car. Its purpose is to measure the contact forces acting on the tires under realistic driving conditions. The contact forces act where the wheel touches the ground, and in general, three components of moment and three components of force may be present at the contact.

The transducer will measure three force components, and three moment components. The transducer doesn't record contact forces directly. Instead, it records the forces and moments exerted by the transducer on the wheel's hub. These readings need to be corrected. Our mission is to derive a formula that can be used to calculate the contact forces from the transducer reading. We'll neglect the wheel's mass, to keep things simple.

This is again a standard statics problem. We know the forces applied on the wheel by the force transducer (because the instrumentation tells us the values). We don't know the contact force, and will calculate it from the condition that the wheel is in static equilibrium

The picture shows forces acting on the wheel, and relevant dimensions. The force transducer records force components $\mathbf{F}^T\mathbf{F}^T$ and moments $\mathbf{M}^T\mathbf{M}^T$ that are exerted by the transducer on the wheel's hub. We have assumed that the transducer forces act on the outside of the wheel hub at the center of the wheel, as shown in the picture. Our goal is to calculate a formula for the contact forces $\mathbf{F}^C$ and moments $\mathbf{M}^C$, in terms of the measured $\mathbf{F}^T$ and $\mathbf{M}^T$, as well as geometrical variables.